THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Suggested solutions to homework 3

If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

5.34. Show that if a normed space has n linearly independent vectors, then so does its dual space.

Solution. Suppose $x_1, x_2, \cdots, x_n$ are linearly independent vectors in X. Then $Z := span\{x_1, \cdots, x_n\}$ is a subspace of X. Define bounded linear functionals $f_1, \cdots, f_n$ on Z by $f_i(x_j) = \delta_{ij}$. By Hahn-Banach theorem, each f_i has an extension $\widetilde{f_i} \in X^*$.

If $\sum_{i=1}^{n} a_i \widetilde{f_i} = 0$ is the zero functional, then $\sum_{i=1}^{n} a_i \widetilde{f_i}(x) = 0, \forall x \in X$. Take $x = x_j \in Z \subset X$ for each $1 \leq j \leq n$ and then

$$\sum_{i=1}^{n} a_i \widetilde{f_i}(x_j) = \sum_{i=1}^{n} a_i \delta_{ij} = a_j = 0, \quad \forall 1 \leq j \ \leq n.$$

Therefore, $\widetilde{f_1}, \cdots, \widetilde{f_n}$ are linearly independent vectors in X^*.

6.5. Let X be a Banach space, Y a normed space, and $T_n : X \to Y$ a sequence of bounded operators such that $\sup\{\|T_n\| : n \in \mathbb{N}\} = \infty$. Show that there exists $x_0 \in X$ such that $\sup\{\|T_n x_0\| : n \in \mathbb{N}\} = \infty$.

Proof. This is an immediate consequence of the Uniform Boundedness Theorem. Otherwise if for every $x \in X$ we have $\sup\{\|T_n x\| : n \in \mathbb{N}\} < \infty$, i.e.,

$$\|T_n x\| \leq c_x, \quad \forall n \in \mathbb{N},$$

where c_x is a real number, then the sequence of the norms $\|T_n\|$ is bounded and contradiction arises.

6.7. Solution.

(a)$\Longrightarrow$(b): There exists $M > 0$ such that $\|T_n\| \leq M, \forall n$. So for every $x \in X$,

$$\|T_n x\| \leq \|T_n\| \|x\| \leq M\|x\|.$$

(b)$\Longrightarrow$(c): For every $x \in X$, there exists $M_x > 0$ such that $\|T_n x\| \leq M_x, \forall n$. Then for every $f \in Y^*$,

$$|f(T_n x)| \leq \|f\| \|T_n x\| \leq M_x \|f\|.$$

(b)$\Longrightarrow$ (a): follow from the Uniform Boundedness Theorem immediately.

(c)$\Longrightarrow$(b): Let us write$T_n x = x_n$ and $f(x_n) = g_n(f)$. Then $(g_n(f))$ is bounded for every $f \in Y*$, so that $(\|g_n\|)$ is bounded by Uniform Boundedness Theorem, and it holds that $\|g_n\| = \|x_n\| = \|T_n x\|$.